I. Volumes of Rotations and Solids with Known Cross Sectio

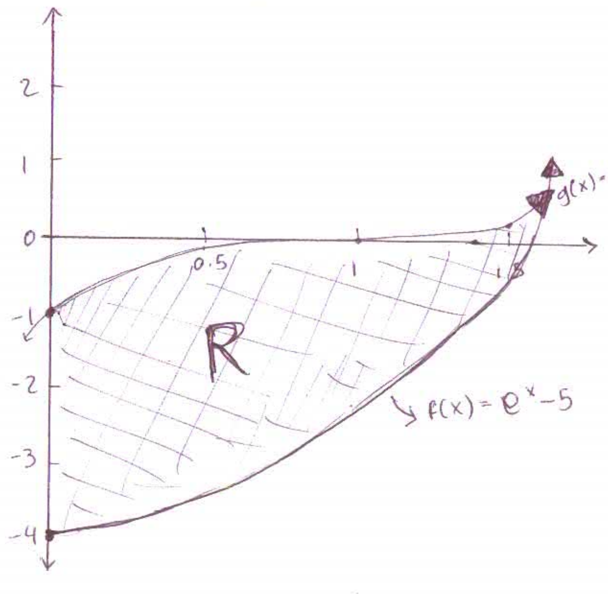
1) Find the volume generated by revolving the region bounded by $y=\sqrt{x}$ and the lines $y=1$, $x=4$ about the line $y=1$

2) Find the volume of the region that is bounded by $y=x^2$ and $y=2x$ and revolved about the line $x=2$.

3) Find the volume of the region that is bounded by $y=\sqrt{x}$, the x-axis and the line $x=4$ revolved about the y-axis.

4) Find the volume of the region bounded by $y=\sqrt{x}$, the x-axis, and the line $x=4$ revolved about the x-axis. (Notice: similar to above.)

5) Suppose R is the region in the plane shown below, bounded by the curves $f(x)=e^x-5$, $g(x)=(x-1)^3$ and the y-axis and R is the base of a solid. If each cross section of the solid is perpendicular to the x-axis is an isosceles right triangle with one leg in the base, what is the volume of the solid?



II. Arc Length and Distance traveled by Particles

1) Find the length of the curve $f(x)=\frac{4\sqrt{2}}{3}x^{3/2}$, $0\le x\le 1$.

2) A portion of a roller coaster track is modeled by the graph of $y=100+30\sin 0.03x+40\cos(0.02x-1)$ from $x=0$ to $x=628$ ft. Find the length of this portion of the track.

3) A particle move along the x-axis with initial position $x(0)=1$. The velocity of the particle at time $t\ge 0$ is given by $v(t)=\sin(t)^{1/2}$

a) What is the distance, of the particle, that is traveled over the time $0\le t\le 15$

b) What is the position of the particle at time $t=15$

PRACTICE AP PROBLEMS

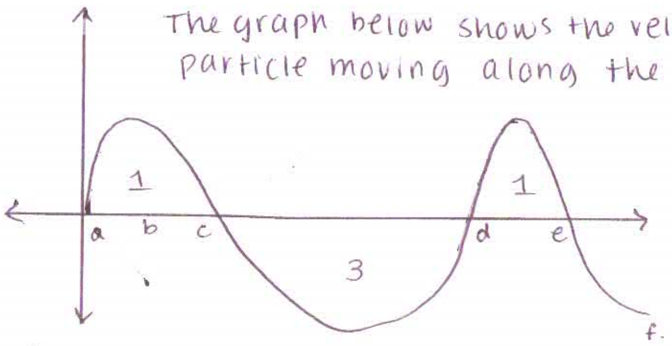
1) Find the volume of the solid of Revolution determined by rotating the area bounded by the graphs of $f(x) = x^2$ and $g(x) = 3x$ about the x-axis. * NO calculator allowed*

   A) $\frac{162\pi}{5}$
   B) $\frac{27\pi}{2}$
   C) 9π
   D) 28π
   E) $\frac{240\pi}{7}$

2) Let R be the region enclosed by the graphs of $f(x) = ax(2-x)$ and $g(x) = ax$ for some positive number a. * No calculator allowed*

   a) Find the area of the region R.
   b) Find the volume when the solid of revolution generated when R is rotated about the x-axis.
   c) Assume a solid exists with a cross section area of R and uniform thickness π. Find the value for a for which the solid has the same volume as the solid in b).

3) The graph below shows the velocity in cm/sec of a particle moving along the y-axis



* Note the numbers in the graph are the areas of the enclosed regions. Figure not drawn to scale!

   i) The particle is at y=1 at the time t=a. what is the particles distance from the origin at t=e

A) y=1 B) y=0 C) y=3 D) y=4 E) y=5

   ii) If the particle travels a total distance of 8 units from time t=a to t=f what is the total distance traveled by the particle from t=e to t=f

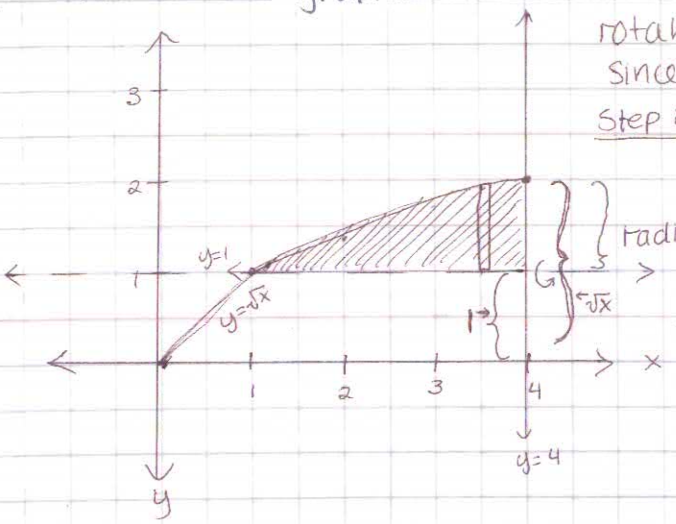
A) 2 B) -7 C) 3 D) -3 E) cannot be determined.

Resources used: Kaplan AB/BC calculus 2007 Edition Calculus Addison-Wesley PaulsOnlineNotes.com

I. Volumes of Rotations etc.

Solution to 1.

Step 1 Draw the graphs and shade the region, indicating the direction of rotation. Identify which method to use:



Since it (area) attatched to axis of rot: Disc metho

Step 2 Identify the radius (distance from axis of rotation)

radius $= \sqrt{x} - 1$

Step 3 Plug the radius into the formula for the Disc method.

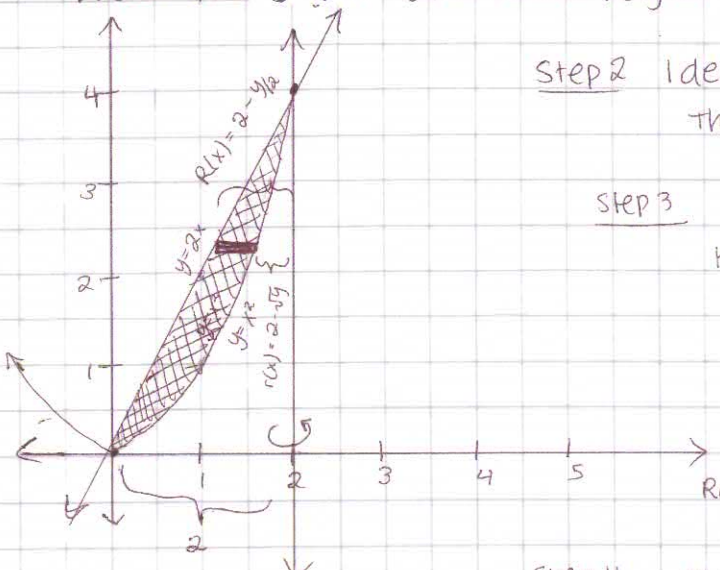
$V = \pi \int_a^b r(x)^2 dx$

$V = \pi \int_1^4 (\sqrt{x} - 1)^2 \, dx$

$V = \boxed{\frac{7\pi}{6}} \approx \boxed{3.665}$

Solution to 2.

Step 1. Draw the graphs and shade region, indicating the direction of rotation



Step 2 Identify which method to use: there is a "hole" ∴ washer method.

Step 3 Idenify the radii r(x) and R(x) knowing that the integral will be in terms of y.

$y = x^2 \rightarrow x = \sqrt{y}$

$y = 2x \rightarrow x = y/2$

Radii: $r(x) = 2 - \sqrt{y}$

$R(x) = 2 - y/2$

Step 4 Plug the radius into the formula for the washer method.

$V = \pi \int_a^b (R(y)^2 - r(y)^2) dy$

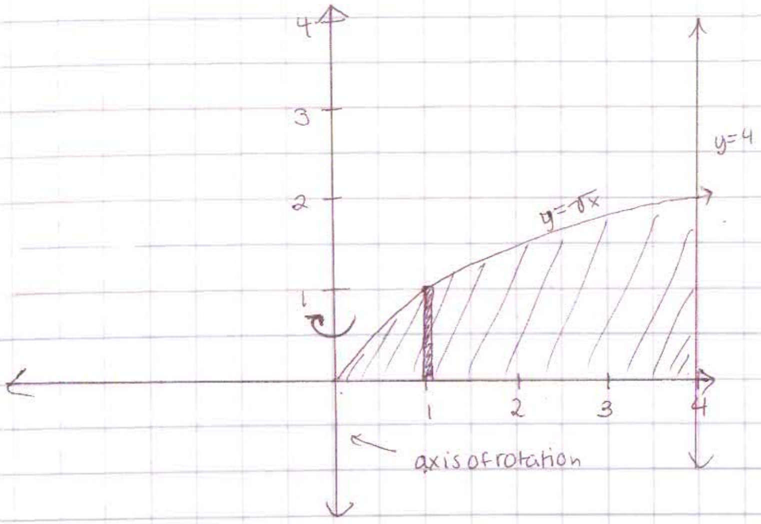
$V = \pi \int_0^4 ((2 - y/2)^2 - (2 - \sqrt{y})^2) \, dy$

$\boxed{V = \frac{8\pi}{3}}$

I. Volumes of R. cont

Solution 3.

Step 1. Sketch the region and indicate the direction of rotation.



Step 2 Identify the easiest method: Shell method

Step 3 Identify radius and height.

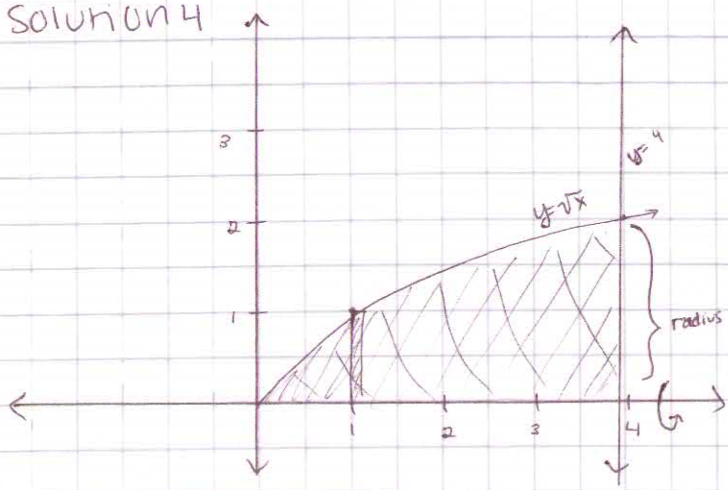
radius = x
height = $\sqrt{x}$

Step 4 Plug into Shell formula.

$$V = 2\pi \int_a^b r(x)\, h(x)\, dx$$

$$V = 2\pi \int_0^4 x\sqrt{x}\, dx = \boxed{\frac{128}{5}\pi} \approx \boxed{80.425}$$

Solution 4



Step 1 Sketch the graph & indicate the direction of rotation.

Step 2 Identify the easiest method: Disc method.

Step 3 Identify the radius:

$r = \sqrt{x}$

Step 4 Plug in Disc formula:

$$V = \pi \int_a^b (r(x))^2 dx$$

$$V = \pi \int_0^4 (\sqrt{x})^2 dx$$

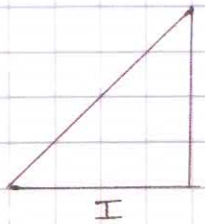
$$V = \boxed{8\pi}$$

I. Vol of rotations cont.

Solution 5.

Step 1: Since the cross section is an isosceles triangle we know the formula for an isoceles triangle is $\frac{1}{2}I^2$ where I is the length of the leg in the base.
If we express area of this cross section as $A(x)$ then its volume between the boundries a&b will be $\int_a^b A(x)\,dx$

Step 2: Draw an image of the triangle



The length of the base $= g(x) - f(x)$
$= (x-1)^3 - (e^x - 5)$
Length of Base = I

∴ Area of the typical cross section
$\frac{1}{2}I^2$
$= \frac{1}{2}((x-1)^3 - (e^x-5))^2$

Step 3: Determine where the graphs intersect to know the limits of integration (a and b)

Using the calculator: 1.6671 is where the graphs intersect.
(x=0 will be 'a' since it is a boundry)

Step 4: Plug into formula.

$$V = \int_0^{1.6671} \frac{1}{2}\left[(x-1)^3 - (e^x-5)\right]^2 dx = \boxed{5.271}$$

II Arc Length...

Solution to 1.

Step 1. Find $f'(x)$

$$f(x) = \frac{4\sqrt{2}}{3} x^{3/2}$$
$$f'(x) = \left(\frac{3}{2}\right)\left(\frac{^{2}4\sqrt{2}}{3}\right) x^{1/2}$$

$$f'(x) = 2\sqrt{2}\, x^{1/2}$$

$\Rightarrow$

Step 2. Plug into arc-length equation.

$$L = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

$$L = \int_0^1 \sqrt{1 + (2\sqrt{2}x^{1/2})^2}\, dx$$

$$L = \boxed{\frac{13}{6}}$$

Solution to 2.

Step 1. Find y'

$$y = 100 + 30\sin(0.03x) + 40\cos(0.02x-1)$$

$$y' = (0.03)(30)(\cos(0.03x)) + (40)(0.02)(-\sin(0.02x-1))$$
$$y' = 0.9\cos(0.03x) - 0.8\sin(0.02x-1)$$

Step 2. Evaluate the arc length from $x = 0$ ft to $x = 628$ ft

$$L = \int_0^{628} \sqrt{1 + (0.9\cos(0.03x) - 0.8\sin(0.02x-1))^2}\, dx$$

$$L = \boxed{804.702 \text{ ft}}$$

The length of this portion of the track is about 804.702 ft

Solution to 3.

a) Step 1 velocity is $\sin\sqrt{t}$ therefore the total distance will be the accumulated change in velocity from $0 \le t \le 15$

Step 2: Plug in Distance $= \int_0^{15} \sin\sqrt{t}\, dt \approx \boxed{4.429}$

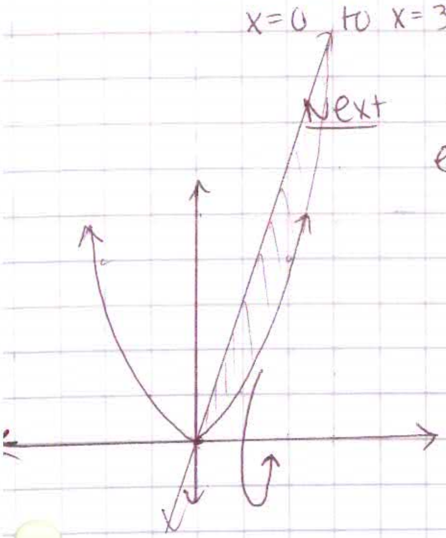
b) Step 1 position at $t=15$ is the change in position over $0 \le t \le 15$ and its initial position. $x(0) = 1$ (Recall*)

Step 2: Add the two values together $x(15) = 1 + 4.429 = \boxed{5.429}$

PRACTICE AP PROBLEMS

Solution to 1.

A First note that $f(x)$ is always below $g(x)$ on the interval $x=0$ to $x=3$ (where the two graphs intersect)



Next Draw a quick sketch; determine the washer method easiest to use and calculate the radii

$R(x) = 3x$ $\qquad V = \pi \int_a^b (R(x)^2 - r(x)^2)\,dx$

$r(x) = x^2$

$$V = \pi \int_0^3 (3x)^2 - (x^2)^2\,dx$$

$$V = \pi \left[3x^3 - \frac{x^5}{5})\right]_0^3$$

$$V = \pi\left(81 - \frac{243}{5}\right) = \boxed{\frac{162\pi}{5}} \Rightarrow A$$

Solution to 2.

Step 1 Find the limits of integration:

a) $f(x) = g(x)$

$ax(2-x) = ax$

when $x=0$ or $x=1$

Therefore the area ~~given~~ of the region is given by:

$$A = \int_0^1 f(x) - g(x)\,dx = \int_0^1 (2ax - ax^2) - ax\,dx = \int_0^1 ax - ax^2\,dx = \frac{ax^2}{2} - \frac{ax^3}{3}$$

$$= \frac{a}{2} - \frac{a}{3} = \boxed{\frac{a}{6}}$$

b) The volume of revolution is given by washer method.

$$V = \pi \int_0^1 R(x)^2 - r(x)^2\,dx = \pi \int_0^1 (2ax - ax^2)^2 - (ax)^2\,dx$$

$$= \pi \int_0^1 4a^2x^2 - 4a^2x^3 + a^2x^4 - a^2x^2\,dx$$

$$= \pi \left(a^2x^3 - a^2x^4 + \frac{a^2x^5}{5}\right)\Big]_0^1 = \pi\left(a^2 - a^2 - \frac{a^2}{5}\right) - 0 = \boxed{\frac{\pi a^2}{5}}$$

c) the volume created by the cross section R with π thickness is $\frac{\pi a}{6}$.

Step 1. Find a positive value for a for which $\frac{\pi a}{6} = \frac{\pi a^2}{5}$ $\quad \therefore a = 5/6$ the volumes are equal

$$\frac{5\pi a}{6\pi a} = \frac{6\pi a^2}{6\pi a}$$

$$\boxed{a = \frac{5}{6}}$$

Practice AP Problems.
Solution to 3

i) B.

Step 1 calculate the distance total.

from $t=a$ to $t=c$ 1 unit upward
from $t=c$ to $t=d$ 3 units downward
from $t=d$ to $t=e$ 1 unit upward
$\therefore$ from $t=a$ to $t=e$ 1 unit downward

Since the initial position was $y=1$ the end position will be $y=0$

ii) C.

Step 1 total distance from $t=a$ to $t=e$ is 5 units
total distance from $t=a$ to $t=f$ is 8 units

$\therefore$ the distance from $t=e$ to $t=5$ is 3units.